

Generalizing Hartogs' Trichotomy Theorem

David Feldman
University of New Hampshire

Mehmet Orhon
University of New Hampshire

with an appendix by

Andreas Blass
University of Michigan

Introduction

The Trichotomy Principle says that for any pair of sets A and B , either a bijection connects the two, or else precisely one set of the pair will inject into the other. Hartogs established the logical equivalence, over **ZF**, between the Trichotomy Principle and the Well-Ordering Principle. As **ZF** suffices to prove the Schröder-Bernstein theorem, the heart of Trichotomy lies in the existence of at least one injection connecting arbitrary A and B (in whichever direction).

Hartogs' theorem seems a remarkable achievement, even judged against the large industry that eventually flowered around the theme of logical equivalents of the Axiom of Choice. Trichotomy certainly follows quickly from the Well-Ordering Principle, but deriving enough structure out of Trichotomy to well-order an arbitrary set might seem daunting.

Hartogs had the remarkable idea of associating to any set A a certain ordinal that we write here as $\omega(A)$. The definition of $\omega(A)$, as we shall now recall it, depends on von Neumann's formulation of ordinals. Von Neumann views ordinals as *hereditarily transitive sets*. By definition, each element of a *transitive* set κ also occurs as a subset of κ , and *hereditarily transitive* means transitive with only transitive elements. An ordinal, viewed as an ordered set, contains as elements just some other ordinals ordered by inclusion (or equivalently, by membership); in fact an ordinal's elements coincide with its proper initial segments. Given any well-ordered set, one may produce (by transfinite induction) a unique ordinal with the same order type.

Hartogs first associates to a set A the set $W(A)$ of *all* well-ordered sets modeled on subsets of A and then defines the ordinal $\omega(A)$ as the set of all ordinals order-isomorphic to elements of $W(A)$. Hartogs argues that

$\omega(A)$ cannot inject into A (lest $\omega(A)$ contain itself). Then Trichotomy guarantees that A injects into $\omega(A)$.

Here we strengthen Hartogs' theorem, by deriving the well-ordering principle from seemingly weaker statements.

We say a family of sets \mathcal{F} *contains an injective* if there exists at least one injective map i which has, for its source and target, distinct sets in \mathcal{F} .

We say the *k-Trichotomy Principle* holds if every family of cardinality k contains an injective; so 2-Trichotomy recaptures classical Trichotomy.

Here we shall prove the

Theorem 1. *For all finite k , the k -Trichotomy Principle implies the Well-Ordering Principle.*

Notation and basic terminology

For sets A and B we write $A \leq B$ if some map injects A into B . We write $A \cong B$ if a bijection takes A to B . We write $A < B$ if $A \leq B$ but not $A \cong B$.

We call a set *infinite* if every finite ordinal injects into it. We write $\omega = \{0, 1, 2, \dots\}$ for the smallest infinite ordinal. We view ordinals as sets of ordinals. In the body of the paper we shall always take *cardinal* in its narrow sense, meaning an ordinal larger than any of its elements. Nevertheless in our opening and closing remarks we use *cardinal* in the broader sense, an equivalence class of sets under the relation of admitting a mutual bijection.

We write $A + B$ for the disjoint union of A and B and nA for the disjoint union of n copies of A .

By a subquotient of a set A , we mean any quotient set of a subset of A .

We write 2^A for the powerset of A .

Proof of the Theorem

In the sequel, a (k) indicates a proposition whose validity depends on k -Trichotomy.

Definition Given any set A , well-orderable or not, we shall write $\omega(A)$ for the smallest ordinal that does not inject into A (as per the Introduction). $\omega(A)$ always exists; it contains precisely the ordinals of all possible well-orderings of subsets of A . For well-orderable A , $A < \omega(A)$; for non-well-orderable A , no injection will connect A and $\omega(A)$ in either direction.

By its definition, $\omega(A)$ gives a strict upper bound on the cardinality of a well-ordered set κ injecting into A . But observe also that an injection from A into $B + \kappa$ hits fewer than $\omega(A)$ elements of κ .

(*k*) **Lemma 1.** *Given a cardinal κ and sets $A_1 \leq \dots \leq A_i \leq \dots \leq A_k$ with $\omega(A_i) = \kappa$ for all i , there exist $n < m$ and a finite set R such that $A_m \leq A_n + R$; in case $\kappa > \omega$, we even take $R = \emptyset$ so that $A_m \cong A_n$.*

Proof. Fix any decreasing sequence of infinite cardinals $\kappa_1 > \dots > \kappa_k = \kappa$. (Work backwards, say, setting each $\kappa_j = \omega(\kappa_{j+1})$.)

Apply k -Trichotomy to the family $A_1 + \kappa_1, A_2 + \kappa_2, \dots, A_k + \kappa_k$ to get an injection $A_m + \kappa_m \rightarrow A_n + \kappa_n$ with m and n distinct.

Focus just now on the restricted injection $i : \kappa_m \rightarrow A_n + \kappa_n$. As $i^{-1}(A_n)$ and $i^{-1}(\kappa_n)$ partition the cardinal κ_m , κ_m admits a bijection with one of these. But if $i^{-1}(A_n)$ bijects with κ_m , κ_m injects into A_n contradicting $\omega(A_n) = \kappa \leq \kappa_m$. So $i^{-1}(\kappa_n)$ bijects with κ_m ; κ_m injects into κ_n ; and we must have $n < m$.

Restrict $A_m + \kappa_m \rightarrow A_n + \kappa_n$ next to $j : A_m \rightarrow A_n + \kappa_n$. $\omega(A_m) = \kappa$ implies $j^{-1}(\kappa_n) < \kappa$. Write $R := j(j^{-1}(\kappa_n))$; so $R \cong j^{-1}(\kappa_n) < \kappa$. Then R also injects into A_n since $\omega(A_n) = \kappa$ too.

$\kappa = \omega$ makes R finite. $\kappa > \omega$ makes ω (and also R) inject into A_n . So $A_n \cong S + R$, say, and also $A_n \cong T + \omega$, say. Now on R infinite (and well-ordered) we have $R + R \cong R$, so $A_n \cong S + R \cong S + R + R \cong A_n + R$. On R finite we have $A_n \cong T + \omega \cong T + \omega + R \cong A_n + R$. Either way, $A_n + R \cong A_n$. Thus A_m injects into A_n . But $A_n \leq A_m$ by hypothesis, so Schröder-Bernstein finally yields $A_n \cong A_m$. \square

Lemma 2. *For an infinite set A , the following are equivalent:*

- (i) *There exists no injective map from ω to A ;*
- (ii) *A bijects with no proper subset of A ;*
- (iii) *$\omega(A) = \omega$.*

As usual, we call any such set A *infinite Dedekind finite*.

Proof. From (iii), which says all finite ordinals inject into A , but ω doesn't, we get (i) (plus the infinitude of A); then (i) and the infinitude of A amounts to (iii).

Given an injection $w : \omega \rightarrow A$, form a non-bijective injection $u : A \rightarrow A$ by setting $u(w(k)) = w(k+1)$ for $k \in \omega$ and having u act as the identity on $A \setminus w(\omega)$. Clearly $A \cong u(A) = A \setminus \{w(0)\}$, a proper subset. So if (i) fails, (ii) does too.

Given a non-bijective injection $u : A \rightarrow A$ pick $a \notin u(A)$. We must have $a, u(a), u(u(a)), u(u(u(a))), \dots, u^k(a), \dots$ all distinct, so map ω to A by sending k to $u^k(a)$. So (i) fails if (ii) does. \square

Dedekind finiteness for A implies Dedekind finiteness for all nA , $n > 0$. (Whenever ω injects into nA , at least one copy of A has an infinite preimage.)

(k) **Lemma 3.** *Infinite Dedekind finite sets do not exist.*

Proof For A infinite Dedekind finite, $\omega(A) = \omega$. Apply Lemma 1 to $A \leq 2A \leq \dots \leq kA$ for an injection $mA \rightarrow nA + R$ with $m > n$ and R finite; then A infinite means R injects into A and $nA + R$ injects properly into mA . Composing two injections, mA injects properly into itself. Dedekind finiteness now fails for mA (Lemma 2(ii)), a contradiction.

(k) **Lemma 4.** *For every infinite set A there exists n such that $nA + nA \cong nA$.*

Proof. By Lemma 3 and Lemma 2(iii), $\omega(A) > \omega$. So apply Lemma 1 to $\{iA\}_{i=1,\dots,k}$ to get $mA \cong nA$ with $m > n$.

Then $nA \cong mA \cong nA + (m - n)A \cong mA + (m - n)A \cong (2m - n)A$, and continuing this way, $nA \cong (m + q(m - n))A$ for all q . Taking q large, we may suppose we had $m > 2n$ in the first place. But $mA \cong nA$ certainly implies $(m + r)A \cong (n + r)A$, so we may even suppose we had $m = 2n$. \square

Remark In **ZF**, as Lindenbaum and Tarski show, $nP \cong nQ$ implies $P \cong Q$ (see Conway and Doyle).

Lemma 5. *If a set X admits a well-ordering, any subquotient Y of X does too.*

Proof. Fix a well-ordering of X . By definition, Y bijects with a family $\{X_y\}_{y \in Y}$ of disjoint subsets of X . Each X_y has a minimal element m_y

according to the order on X . By the disjointness of the X_y , Y bijects with $\{m_y\}$, but $\{m_y\}$ has a well-ordering as a subset of X . \square

Lemma 6. *A set A admits a well-ordering if $A \cong A + A$ and there exists an injection $k : 2^A \rightarrow A + \kappa$ for some ordinal κ .*

Proof. We aim to exhibit A as a subquotient of κ (Lemma 5).

Write $A = B + C$, with $A \cong B \cong C$; exhibiting C as a subquotient of κ suffices.

For $c \in C$, define $S_c := \{X \subseteq A \mid X \cap C = \{c\}\}$. Naturally $S_c \cong 2^B \cong 2^A$. Certainly $S_c \cap S_d = \emptyset$ unless $c = d$. Also $T_c := k(S_c) \cap \kappa \neq \emptyset$ since $k(S_c) \cong 2^A > A$ (Cantor). Therefore $\{T_c\}_{c \in C}$ forms a collection of pairwise disjoint nonempty sets in κ . \square

Proof of the Main Theorem: Fix any set S . Set $A = nS$ with n so large that $A + A \cong A$ (Lemma 4).

Write $\mathcal{P}^0(A) = A$, $\mathcal{P}^{i+1}(A) = 2^{\mathcal{P}^i(A)}$ and $\omega^1(A) = \omega(A)$, $\omega^{i+1}(A) = \omega(\omega^i(A))$.

k -Trichotomy for the family $\{\mathcal{P}^i(A) + \omega^{k-i}(\mathcal{P}^{k-1}(A))\}_{i=0,\dots,k-1}$ yields an injection

$$\mathcal{P}^i(A) + \omega^{k-i}(\mathcal{P}^{k-1}(A)) \rightarrow \mathcal{P}^j(A) + \omega^{k-j}(\mathcal{P}^{k-1}(A))$$

which in turn restricts to an injection

$$\omega^{k-i}(\mathcal{P}^{k-1}(A)) \rightarrow \mathcal{P}^j(A) + \omega^{k-j}(\mathcal{P}^{k-1}(A))$$

that possesses a (well-ordered) cardinal for its domain. As such we obtain the existence of still another injection, either of the form

$$\omega^{k-i}(\mathcal{P}^{k-1}(A)) \rightarrow \mathcal{P}^j(A)$$

or

$$\omega^{k-i}(\mathcal{P}^{k-1}(A)) \rightarrow \omega^{k-j}(\mathcal{P}^{k-1}(A)).$$

Whereas Hartogs' original argument actually rules out an injection of former sort (since $j \leq k-1$), the existence of an injection of the latter sort now forces $i > j$.

The original injection also restricts to

$$\mathcal{P}^i(A) \rightarrow \mathcal{P}^j(A) + \omega^{k-j}(\mathcal{P}^{k-1}(A)) .$$

so we get an injection

$$\mathcal{P}^i(A) \rightarrow \mathcal{P}^{i-1}(A) + \omega^{k-j}(\mathcal{P}^{k-1}(A)) .$$

Thus $\mathcal{P}^{i-1}(A)$ satisfies the hypothesis of Lemma 6, so admits a well-ordering. But $A < 2^A$ (by singletons) and then $A < \mathcal{P}^{i-1}(A)$ (by induction), so a well-ordering of $\mathcal{P}^{i-1}(A)$ induces a well-ordering of A which induces a well-ordering of S .

Remark One may read our main theorem to say that the existence of a pair of incomparable cardinals entails the existence of a family of k mutually incomparable cardinals. Amidst two incomparable cardinals, at least one cardinal admits no well-ordering. Blass' appendix below offers a direct construction of a family of k incomparable cardinals starting from any single set that admits no well-ordering.

Remark We can find no obvious modification of our methods to show that ω -trichotomy (meaning that any countable family of cardinals contains at least one comparable pair) implies AC. While we hope to return to this matter, for now we merely record a few observations.

First, one must distinguish between ω -trichotomy and ∞ -trichotomy (meaning that any infinite family of cardinals contains at least one comparable pair). Certainly ∞ -trichotomy immediately implies ω -trichotomy, but the converse remains unclear on account of the possibility of an infinite Dedekind finite family of incomparable cardinals.

Combinatorially (if not set-theoretically), ω -trichotomy actually turns out stronger than it sounds. Recall that a special case of Ramsey's theorem guarantees that any two-colored complete graph on a countably infinite set of vertices possesses a monochromatic subgraph. Now, given a countably infinite family of cardinals, we may regard these as the vertices of a complete graph, taking *comparability* and *non-comparability* as the colors. Ramsey's theorem then promises either an infinite subfamily of mutually comparable cardinals or an infinite subfamily of mutually incomparable cardinals; as ω -trichotomy rules out the latter, ω -trichotomy guarantees that every countably infinite family of cardinals contains an infinite chain.

We get even more on account of the effective nature of certain proofs of Ramsey's theorem. Recall how the proof can run. Call the two colors c_1 and c_2 and write $c(v_i, v_j)$ for the color of the edge connecting vertices v_i and v_j . Filter the vertex set successively in order to produce an infinite subgraph in which $c = c(v_i, v_j)$ depends only on i at least for $j > i$ and assign vertex v_i color c . In the end, we will have assigned at least one of the two colors to infinitely many of the vertices that survive. If we have used *exactly* one of the colors infinitely often, we pass to the subset of vertices that received that color; otherwise we

pass to the subset of vertices that received color c_1 . The proof thus explicitly *canonizes* a particular monochromatic subgraph of our graph.

Combining the results of the previous two paragraphs, from a given countably infinite family of cardinals, if we have ω -trichotomy, we can isolate within the family a canonical infinite chain. Now we can remove that chain and find another. Indeed, by induction, ω -trichotomy implies that we may partition any countably family of distinct cardinals into infinite chains (and perhaps a finite residual). (Observe that the induction does not depend upon AC because we remove a well-determined chain at each step.) If we like, we can continue applying ω -trichotomy to cross-sections of such a partition to adduce still more structure.

Finally, we describe a very simple topos example as weak evidence against the provability of AC from ω -trichotomy. Consider the topos of sheaves of (**ZFC**) sets over a two point discrete space. A (global) injection in this topos means a set injection at each stalk. View the elements of this topos simply as ordered pairs of sets, so one has (κ_1, κ_2) incomparable with (λ_1, λ_2) if $\kappa_1 > \lambda_1$ and $\kappa_2 < \lambda_2$ or vice-versa. Certainly finite anti-chains of any length exist. An infinite anti-chain, however, would entail an infinite *descending* chain of cardinals in one coordinate corresponding to any infinite *ascending* chain of cardinal in the other, so impossible. Simplicity notwithstanding, this example shows that no sufficiently constructive method can extrapolate, from counterexamples to k -trichotomy for all finite k , to a counterexample to ω -trichotomy. Perhaps a suitable construction, first with ur-elements and then with forcing, can exploit this same device in the context of classical models of set theory.

Appendix by Andreas Blass

For any set X , define $\mathcal{Q}(X) := X \times \mathcal{P}(X)$ and $\mathcal{Q}^j(X) := \mathcal{Q}(\mathcal{Q}^{j-1}(X))$. Because of the injection $X \rightarrow \mathcal{Q}(X)$ sending any x to (x, \emptyset) , we have $X \leq \mathcal{Q}(X)$ and therefore, by induction, $X \leq \mathcal{Q}^j(X)$ for all j .

Lemma 7. *For a set X and an ordinal κ , an injection $\theta : \mathcal{Q}(X) \rightarrow X + \kappa$ induces a canonical well-ordering of X .*

Proof. For $x \in X$, Cantor's theorem prevents $\theta(\{x\} \times \mathcal{P}(X))$ from being included in X . Thus, as x varies through X , the sets $T_x = \kappa \cap \theta(\{x\} \times \mathcal{P}(X))$ are nonempty, pairwise disjoint subsets of κ . This exhibits X as a subquotient of κ , so Lemma 5 provides a well-ordering of X . \square

For set X and integer $k > 1$, define well-ordered cardinals

$$\kappa_0(X, k), \dots, \kappa_{k-1}(X, k)$$

by the recursion

$$\kappa_0(X, k) := \omega(\mathcal{Q}^{k-1}(X)) \quad \text{and} \quad \kappa_{i+1}(X, k) = \omega(\mathcal{Q}^{k-i-1}(X) + \kappa_i(X, k)).$$

For X and k fixed, the sequence of cardinals $\kappa_i := \kappa_i(X, k)$ increases strictly.

Claim: Whenever X carries no well-ordering, $\{\mathcal{Q}^{k-i-1}(X) + \kappa_i(X, k)\}$ constitutes an explicit example of k pairwise incomparable sets.

If $\{\mathcal{Q}^{k-i-1}(X) + \kappa_i\}$ doesn't violate k -Trichotomy, we must have an injection of the form $\mathcal{Q}^{k-i-1}(X) + \kappa_i \rightarrow \mathcal{Q}^{k-j-1}(X) + \kappa_j$, with $i \neq j$.

In particular, $\kappa_i \leq \mathcal{Q}^{k-j-1}(X) + \kappa_j$, and so, by definition of κ_{j+1} , we have $\kappa_i < \kappa_{j+1}$. From this and $i \neq j$, we get that $i < j$.

Now writing $Y := \mathcal{Q}^{k-j-1}(X)$, we have

$$\mathcal{Q}^{j-i}(Y) = \mathcal{Q}^{k-i-1}(X) \leq \mathcal{Q}^{k-i-1}(X) + \kappa_i \leq \mathcal{Q}^{k-j-1}(X) + \kappa_j \cong Y + \kappa_j.$$

Since $j > i$, we obtain an injection $\theta : \mathcal{Q}(Y) \rightarrow Y + \kappa_j$. Lemma 7 now makes Y well-orderable, and X too, because $X \leq Y$.

REFERENCES

1. *Set theory and the continuum hypothesis*. W. A. Benjamin, Inc., New York-Amsterdam 1966.
2. Conway, J.H., Doyle, P., Division by Three.
<http://www.math.dartmouth.edu/~doyle/docs/three/three.pdf>
3. L. Gillman, Two classical surprises concerning the axiom of choice and the continuum hypothesis. *Amer. Math. Monthly* 109 (2002), no. 6, 544–553.
4. F. Hartogs, Über das Problem der Wohlordnung. *Math. Ann.* 76 (1915) 438–443.
5. Rubin, Herman; Rubin, Jean E. *Equivalents of the axiom of choice*. North-Holland Publishing Co., Amsterdam 1963.
6. Rubin, Herman; Rubin, Jean E. *Equivalents of the axiom of choice. II*. Studies in Logic and the Foundations of Mathematics, 116. North-Holland Publishing Co., Amsterdam, 1985.
7. W. Sierpinski, *Cardinal and Ordinal Numbers*, 2nd ed. Monografie Matematyczne 32, PWN-Polish Scientific Publishers, Warsaw, 1965.